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SUPERPOSITION TECHNIQUE FOR POTENTIAL FLOW AROUND AN AEROFOIL AND CONTROL OF THE CIRCULATION

M. ADİL YÜKSELEN

Faculty of Aeronautics and Aerospace, İstanbul Technical University, Maslak, 80626 İstanbul, Turkey

SUMMARY

In this paper the superposition technique for a potential flow around an aerofoil is investigated in the complex plane. The control of the circulation around the aerofoil by satisfying the Kutta condition at the flow field points is described.

KEY WORDS: potential flow; panel method; superposition technique

1. INTRODUCTION

Potential flow calculations are an important part of a potential flow/boundary layer coupling method for an aerofoil.[1,2] These calculations are repeated together with the boundary layer calculations until a convergence is obtained. If a complete analysis or design problem is considered at several angles of attack, the numbers of iterations and potential flow calculations may amount to several hundred. Therefore the speed and accuracy of a coupling method depend on the speed and accuracy of the potential flow calculation method used.

The incompressible potential flow around an aerofoil is usually calculated either by a conformal mapping method or by a panel method. The panel methods try to solve some integral forms instead of the governing Laplace equation. For this purpose the integral equation is converted to a system of linear equations by dividing the aerofoil surface into small panels. The coefficient matrix of the resulting system of equations depends only on the aerofoil geometry, while the right-hand side depends on the free flow conditions.[3,4]

A panel method can be applied directly at any angle of attack for an aerofoil. However, a superposition technique is useful if several angles of attack are considered. With this technique, first the solutions for the non-circulating flows around the aerofoil in the uniform flows parallel and vertical to the chord and secondly the solution for the flow around the same aerofoil in a circulating free flow are obtained. Then the complete solutions at different angles of attack are obtained easily by superposing these simple flow solutions. Furthermore, the simple flows are solved simultaneously, since their coefficient matrices are the same. In conclusion, the superposition technique saves on computing time compared with a direct solution.

This classicial superposition technique is not only important when different angles of attack are considered, but may also be advantageous for some coupling techniques using a panel method for potential flow calculations. If the coupling method does not change the aerofoil geometry for calculating the equivalent inviscid flow (e.g. by representing the boundary layer with a suction or

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blowing on the aerofoil surface), the coefficient matrix does not change during all the iteration steps. Thus the calculation of the inverse of this matrix once at the very beginning of the coupling procedure is sufficient. This also saves on computing time. However, this easy procedure is possible only if a superposition technique is used, since the circulation around the aerofoil is changed at each step of the iteration depending on the boundary layer calculation results.

Although the superposition technique in panel methods is used in some computer codes, the detail of the technique seldom appears in the literature. Therefore in this paper the logic and a formulation of a superposition technique are given. The control of the circulation around an aerofoil by applying the Kutta condition at the flow field points (not on the aerofoil surface) is described. The complex plane is preferred because of the ease of formulation. For this purpose a panel method developed in the complex plane by Yükselen[5] is taken as the basic panel method.

BASIC PANEL METHOD

The complex conjugate velocity at any point z of the flow field around an aerofoil in the complex plane (Figure 1) can be written as

$$w^*(z) = w_\infty^* + \frac{1}{2\pi}\oint_C \frac{v^*(z_0)t^*(z_0)}{z - z_0}\,dz_0, \tag{1}$$

where

$$w_\infty = V_\infty e^{i\alpha} = V_\infty \cos\alpha + iV_\infty \sin\alpha, \tag{2}$$

$$t(z_0) = e^{i\delta(z_0)}, \tag{3}$$

$$v(z_0) = \sigma(z_0) + i\gamma(z_0). \tag{4}$$

Here V_∞ and α are the free flow velocity and the angle of attack respectively, z_0 represents the aerofoil surface points, δ is the slope angle at these points, σ and γ are the source and vortex strengths at the surface points respectively and a superscript asterisk indicates the conjugate of any complex variable.[5]

Analytical calculation of the integral in equation (1) is nearly impossible, since the analytical representation of the surface shapes of the aerofoils and the singularity distributions is usually not possible. Therefore in any panel method the aerofoil surface is divided into small panels (Figure 2). The integrals at each panel are calculated separately by making appropriate assumptions on the panel

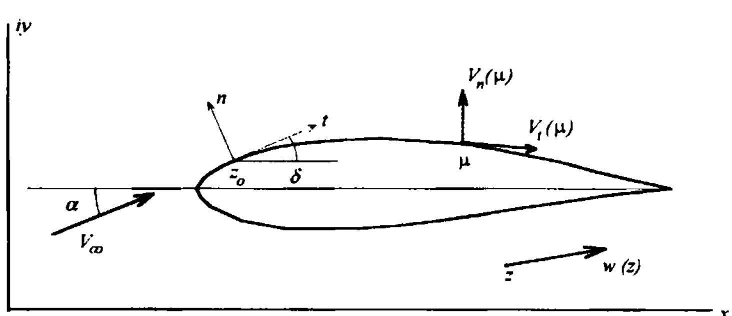

Figure 1. Flow field in complex plane

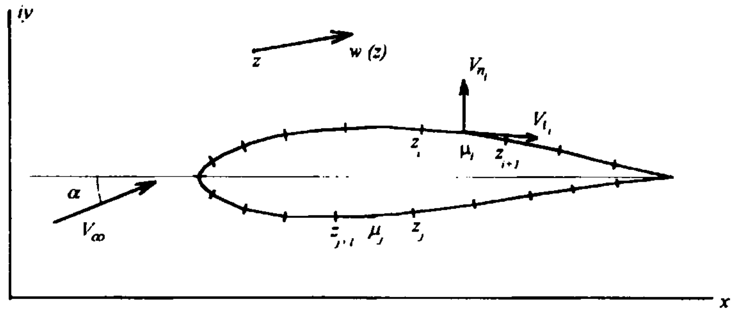

Figure 2. Surface panels

geometry and the singularity distributions. Thus the complex velocity at point z is written in terms of some complex coefficients C_{ij} and unknown complex singularity strengths v_j as

$$w(z) = w_\infty + \sum_{j=1}^{N} C_{ij}^* v_j, \tag{5}$$

where N is the number of panels. The coefficients C_{ij} are dependent on the co-ordinates of end points and the control point of each panel, the panel geometry and the singularity distribution at each panel. The coefficients for linear singularity distribution along straight line panels are given in Reference 5.

In a co-ordinate system based on the tangential and normal directions of the aerofoil surface at any μ_i control point, equation (5) can be written as

$$w_{\mathrm{TN}_i} = V_{\mathrm{T}_i} + \mathrm{i}V_{\mathrm{N}_i} = w_\infty t_i^* + \sum_{j=1}^{N} \bar{C}_{ij}^* v_j, \tag{6}$$

where $\bar{C}_{ij}^* = t_j^* C_{ij}^*$ and V_{T_i} and V_{N_i} are the tangential and normal velocity components respectively. If there is a suction (or blowing) on the aerofoil surface representing the boundary layer, the surface boundary condition at each control point gives the system of equations

$$\sum_{j=1}^{N} \mathrm{Im}\{\bar{C}_{ij}^*(\sigma_j + \mathrm{i}\gamma_j)\} = U_{\mathrm{N}_i} - \mathrm{Im}\{w_\infty t_i^*\} \quad (i = 1, 2, \ldots, N), \tag{7}$$

where U_{N_i} is the suction velocity. If there is no suction (solid boundary condition), $U_{N_i} = 0$.

The number of unknowns in the system of equations (7) is twice the number of equations. Therefore additional assumptions are necessary to close the system. An effective approximation is to take a parabolic trapezoidal vortex distribution (linear on each panel but parabolic along the aerofoil surface; Figure 3). Thus the vortex strength at the control point of any panel is

$$\gamma_j = d_j \gamma_{\mathrm{c}}, \tag{8}$$

where the coefficients of the parabolic distribution, d_j are defined in terms of the distance s on the aerofoil surface[5] as

$$d_j = 0{\cdot}5[\bar{s}_j(\bar{s}_j - 1) + \bar{s}_{j+1}(\bar{s}_{j+1} - 1)], \qquad \bar{s}_j = s_j/s_{\mathrm{T}}. \tag{9}$$

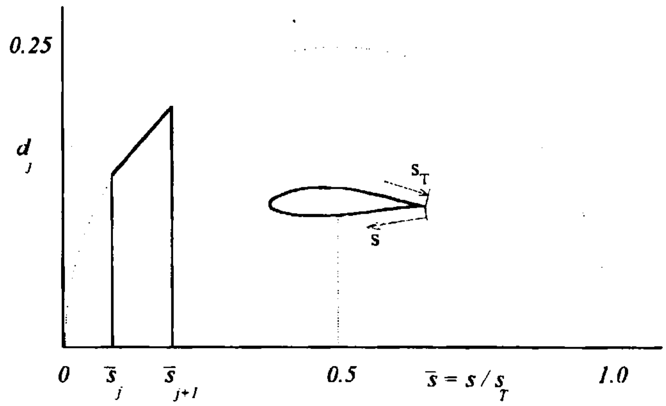

Figure 3. Parabolic tranezoidal vortex distribution

Equations (6) and (7) then become

$$w_{\mathrm{TN}_i} = w_\infty t_i^* + \sum_{j=1}^{N} \bar{C}_{ij}^* \sigma_j + \left(\sum_{j=1}^{N} i\bar{C}_{ij}^* d_j \right) \gamma_c, \tag{10}$$

$$\sum_{j=1}^{N} \mathrm{Im}\{\bar{C}_{ij}^*\} \sigma_j + \mathrm{Im}\left\{ \sum_{j=1}^{N} i\bar{C}_{ij}^* d_j \right\} \gamma_c = U_{\mathrm{N}_i} - \mathrm{Im}\{w_\infty t_i^*\} \quad (i = 1, 2, \ldots, N). \tag{11}$$

There is still an extra unknown in the system of equations (11). An additional equation is obtained from the well-known Kutta condition. As a simple application of this condition for the cases not containing any suction on the surface, the tangential velocities at the control points of two neighbouring panels of the trailing edge are assumed equal, i.e.

$$V_{\mathrm{T}_1} = -V_{\mathrm{T}_N}, \tag{12}$$

where the negative sign is due to the integral direction on the aerofoil.

For the potential flow/boundary layer coupling techniques representing the boundary layer with a suction, the application of the Kutta condition is sometimes preferred at the points occurring in the flow field (on the displacement surface, for example).

SUPERPOSITION TECNIQUE

Since the Laplace equation is linear, simple flow solutions can be superposed to obtain more complicated potential flow fields. This means a summation of the complex potential functions of simple flows in the complex plane to obtain the solution for a more complex flow:

$$f(z) = f^{(1)}(z) + f^{(2)}(z) + f^{(3)}(z) + \cdots. \tag{13}$$

Taking the derivative and then the conjugate of this last equation, a similar relation is obtained for the complex velocities:

$$w(z) = w^{(1)}(z) + w^{(2)}(z) + w^{(3)}(z) + \cdots. \tag{14}$$

On the other hand, recalling equation (2) for the free flow velocity, equation (10) can be written as

$$w_{\mathrm{TN}_i} = V_\infty \cos\alpha\, t_i^* + \mathrm{i}V_\infty \sin\alpha\, t_i^* + \sum_{j=1}^{N} \bar{C}_{ij}^* \sigma_j + \left(\sum_{j=1}^{N} \mathrm{i}\bar{C}_{ij}^* d_j\right)\gamma_c. \tag{15}$$

The third term on the right-hand side of this equation depends only on the aerofoil geometry and the source strengths, while the other terms depend on the free flow conditions and the circulation around the aerofoil (i.e. vortex strength). Thus it is very convenient to write the complex velocity as the summation of some simple flows as follows:

$$w_{\mathrm{TN}_i} = w_i^{(1)} + w_i^{(2)} + w_i^{(3)} + \dots, \tag{16}$$

where

$$w_i^{(1)} = \sum_{j=1}^{N} \bar{C}_{ij}^* \sigma_j^{(1)} + V_\infty \cos\alpha\, t_i^*, \tag{17a}$$

$$w_i^{(2)} = \sum_{j=1}^{N} \bar{C}_{ij}^* \sigma_j^{(2)} + \mathrm{i}V_\infty \sin\alpha\, t_i^*, \tag{17b}$$

$$w_i^{(3)} = \sum_{j=1}^{N} \bar{C}_{ij}^* \sigma_j^{(3)} + \left(\sum_{j=1}^{N} \mathrm{i}\bar{C}_{ij}^* d_j\right)\gamma_c. \tag{17c}$$

The first two equations represent the non-circulating flows around the aerofoil in a uniform parallel flow, with a velocity $V_\infty \cos\alpha$ parallel to the chord and a velocity $V_\infty \sin\alpha$ vertical to the chord respectively. The third equation is for the circulating flow of strength γ_c (Figure 4).

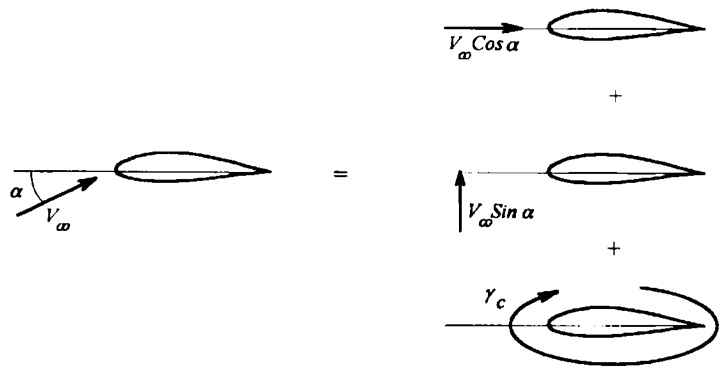

Figure 4. Superposition of simple flows

These equations can be made independent of the free flow conditions and the circulation by dividing the two sides by related values:

$$\bar{w}_i^{(1)} = \frac{w_i^{(1)}}{V_\infty \cos\alpha} = \sum_{j=1}^{N} \bar{C}_{ij}^{*}\bar{\sigma}_j^{(1)} + t_i^{*}, \tag{18a}$$

$$\bar{w}_i^{(2)} = \frac{w_i^{(2)}}{V_\infty \sin\alpha} = \sum_{j=1}^{N} \bar{C}_{ij}^{*}\bar{\sigma}_j^{(2)} + \mathrm{i}t_i^{*}, \tag{18b}$$

$$\bar{w}_i^{(3)} = \frac{w_i^{(3)}}{\gamma_c} = \sum_{j=1}^{N} \bar{C}_{ij}^{*}\bar{\sigma}_j^{(3)} + \left(\sum_{j=1}^{N} \mathrm{i}\bar{C}_{ij}^{*}d_j\right). \tag{18c}$$

Here the source strengths are also non-dimensional:

$$\bar{\sigma}_j^{(1)} = \frac{\sigma_j^{(1)}}{V_\infty \cos\alpha}, \qquad \bar{\sigma}_j^{(2)} = \frac{\sigma_j^{(2)}}{V_\infty \sin\alpha}, \qquad \bar{\sigma}_j^{(3)} = \frac{\sigma_j^{(3)}}{\gamma_c}. \tag{19}$$

Applying the surface boundary condition for each of the simple flows, the following system of equations is obtained:

$$\sum_{j=1}^{N} \mathrm{Im}\{\bar{C}_{ij}^{*}\}\bar{\sigma}_j^{(1)} = -\mathrm{Im}\{t_i^{*}\} + U_{\mathrm{N}_i} \quad (i = 1, 2, \ldots, N), \tag{20a}$$

$$\sum_{j=1}^{N} \mathrm{Im}\{\bar{C}_{ij}^{*}\}\bar{\sigma}_j^{(2)} = -\mathrm{Im}\{\mathrm{i}t_i^{*}\} \quad (i = 1, 2, \ldots, N), \tag{20b}$$

$$\sum_{j=1}^{N} \mathrm{Im}\{\bar{C}_{ij}^{*}\}\bar{\sigma}_j^{(1)} = -\mathrm{Im}\left\{\sum_{j=1}^{N} \mathrm{i}\bar{C}_{ij}^{*}d_j\right\} \quad (i = 1, 2, \ldots, N). \tag{20c}$$

The most important feature of these equations is of course their complete independence of the free flow conditions. Furthermore, this system of equations can be solved simultaneously by calculating the inverse of only one of the influence coefficient matrices, since the coefficient matrix of each system is the same.

The tangential velocities for the simple flows and the superposed flow are then

$$\bar{V}_{\mathrm{T}_i}^{(1)} = \frac{V_{\mathrm{T}_i}^{(1)}}{V_\infty \cos\alpha} = \sum_{j=1}^{N} \mathrm{Re}\{\bar{C}_{ij}^{*}\}\bar{\sigma}_j^{(1)} + \mathrm{Re}\{t_i^{*}\}, \tag{21a}$$

$$\bar{V}_{\mathrm{T}_i}^{(2)} = \frac{V_{\mathrm{T}_i}^{(2)}}{V_\infty \sin\alpha} = \sum_{j=1}^{N} Re\{\bar{C}_{ij}^{*}\}\bar{\sigma}_j^{(2)} + \mathrm{Re}\{\mathrm{i}t_i^{*}\}, \tag{21b}$$

$$\bar{V}_{\mathrm{T}_i}^{(3)} = \frac{V_{\mathrm{T}_i}^{(3)}}{\gamma_c} = \sum_{j=1}^{N} \mathrm{Re}\{\bar{C}_{ij}^{*}\}\bar{\sigma}_j^{(3)} + \mathrm{Re}\left\{\sum_{j=1}^{N} \mathrm{i}\bar{C}_{ij}^{*}d_j\right\}, \tag{21c}$$

$$V_{\mathrm{T}_i} = \bar{V}_{\mathrm{T}_i}^{(1)} V_\infty \cos\alpha + \bar{V}_{\mathrm{T}_i}^{(2)} V_\infty \sin\alpha + \bar{V}_{\mathrm{T}_i}^{(3)}\gamma_c. \tag{22}$$

However, for the last equation we need a value for γ_c. The value of γ_c is calculated by applying the Kutta condition. If the tangential velocities at the control points of neighbouring panels of the trailing edge are made equal, this value is

$$\gamma_c = \frac{(\bar{V}_{T_1}^{(1)} + \bar{V}_{T_N}^{(1)})V_\infty \cos\alpha + (\bar{V}_{T_1}^{(2)} + \bar{V}_{T_N}^{(2)})V_\infty \sin\alpha}{\bar{V}_{T_1}^{(3)} + \bar{V}_{T_N}^{(3)}}. \tag{23}$$

GENERAL APPLICATION OF THE KUTTA CONDITION

The application of the Kutta condition by equating the tangential velocity component on neighbouring panel control points of the trailing edge, as seen in the previous sections, is easy if there is no suction or blowing on the aerofoil surface. For the suction case, if the Kutta condition is applied at the flow field points (e.g. at the displacement surface points), a special investigation is necessary, since the directions of the flow at these points are not known.

Taking Z_U and Z_L as the Kutta condition control points at the upper and lower surfaces of the aerofoil, W_U and W_L are the complex velocities and q_U and q_L are the resulting velocities at these points respectively (Figure 5). The Kutta condition is then

$$q_U = q_L \tag{24}$$

or, since $q = |w| = \surd(ww^*)$,

$$\surd(w_U w_U^*) = \surd(w_L w_L^*). \tag{25}$$

The equality is correct also for the squares of the two sides of this relation.

On the other hand, recalling the superposition technique described in the previous section, the complex velocity at any point of the flow field can be written as a superposition of some simple flows as follows:

$$w = \bar{w}^{(1)}V_\infty \cos\alpha + \bar{w}^{(2)}V_\infty \sin\alpha + \bar{w}^{(3)}\gamma_C \tag{26}$$

or

$$w = \bar{w} + \bar{w}^{(3)}\gamma_C, \tag{27}$$

where

$$\bar{w} = \bar{w}^{(1)}V_\infty \cos\alpha + \bar{w}^{(2)}V_\infty \sin\alpha. \tag{28}$$

The square of the velocity is then

$$q^2 = ww^* = \bar{q}^2 + 2\mathrm{Re}\{\bar{w}(\bar{w}^{(3)})^*\}\gamma_C + (\bar{q}^{(3)})^2\gamma_C^2. \tag{29}$$

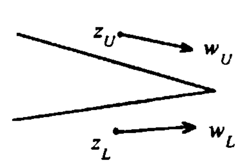

Figure 5. Kutta condition control points

Applying the Kutta condition under these conditions, the following second-degree equation is obtained:

$$A\gamma_C^2 + 2B\gamma_C + C = 0. \tag{30}$$

The positive root of this equation is

$$\gamma_C = [-B + \surd(B^2 - AC)]/A, \tag{31a}$$

where

$$A = (\bar{q}_U^{(3)})^2 - (\bar{q}_L^{(3)})^2, \tag{31b}$$

$$B = Re\{\tilde{w}_U(\bar{w}_U^{(3)})^*\} - \mathrm{Re}\{\tilde{w}_L(\bar{w}_L^{(3)})^*\}, \tag{31c}$$

$$C = (\tilde{q}_U)^2 - (\tilde{q}_L)^2 \tag{31d}$$

and $\tilde{q}$ and $\bar{q}$ are the speeds in the simple flows.

RESULTS AND CONCLUSIONS

The formulation proposed above for the superposition technique was tested on several Karman–Trefftz and Joukowsky aerofoils.[6] Examples are given in Figure 6 and 7 respectively for a symmetrical Joukowsky aerofoil with a thickness ratio of 0·10 and for a 5% cambered Karman–Trefftz aerofoil with a thickness ratio of 0·15. The results for an angle of attack of 4° coincide quite well with the analytical results.

Another example, for an NACA 4412 aerofoil at 4° angle of attack, is given in Figure 8, as compared with the direct application of the complex panel method. Numerical results of the two applications are exactly the same.

The technique for the application of the Kutta condition at the flow field points was also tested on an NACA 0012 symmetrical aerofoil at 4° angle of attack. For this application the superposition technique for the complex panel method is coupled with a boundary layer code based on the Cebeci–Smith method.[7] In Figure 9 the results obtained by the potential flow calculation and the viscous/inviscid coupling method after 24 iterations are given.

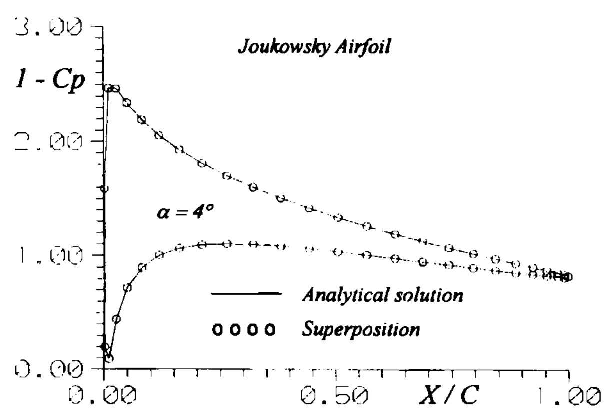

Figure 6. Application of superposition technique on a Joukowsky aerofoil

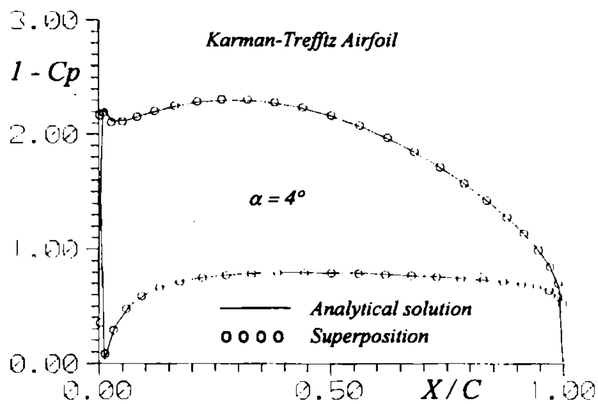

Figure 7. Application of superposition technique on a Karman–Trefftz aerofoil

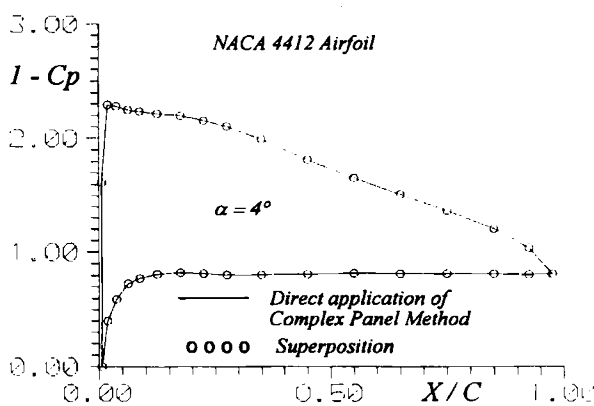

Figure 8. Application of superposition technique on an NACA 4412 aerofoil

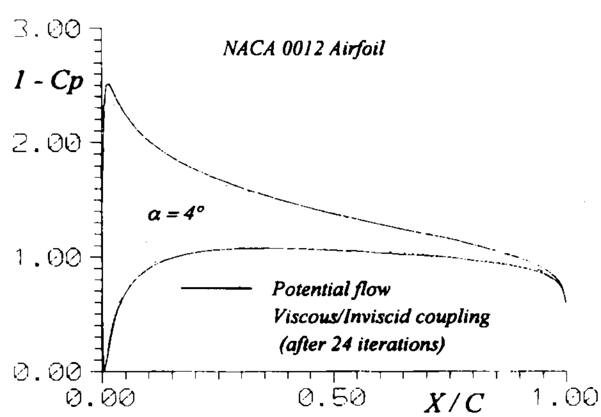

Figure 9. Application of superposition technique in a coupling method

All these example applications show that the formulation given for the superposition technique works correctly and effectively.

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REFERENCES

1. B. R. Williams and R. C. Lock, 'Viscous–inviscid interactions in external aerodynamics', *Prog. Aerosp. Sci.*, **24**, 51–171 (1987).
2. T. Cebeci, R. W. Clark, K. C. Chang, N. D. Halsey and K. Lee. 'Airfoils with separation and the resulting wakes', *J. Fluid Mech.*, **163**, 323–347.
3. J. Katz and A. Plotkin, *Low Speed Aerodynamics, From Wing Theory to Panel Methods*, McGraw-Hill, New York, 1991.
4. J. Bousquet, *Methodes des Singularités*, Editions ENSAE, 1986.
5. M. A. Yükselen, 'The surface singularity method in complex plane', *Proc. Int. Conf. on Computational Mechanics (ICCM86)*, Tokyo, May 1986.
6. M. A. Yükselen and M. Z. Erim, 'A general iterative method to design Karman–Trefftz and Joukowsky airfoils', *Int. j. numer. methods eng.*, **20**, 1349–1368 (1984).
7. T. Cebeci and A. M. O. Smith, *Analysis of Turbulent Boundary Layers*, Academic Press, New York, 1974.